In many interesting situations in astrophysical GR, the source of the gravitational field can be taken to be a perfect fluid as a first approximation. In general, a ‘fluid’ is a special kind of *continuum*. A continuum is a collection of particles so numerous that the dynamics of individual particles cannot be followed, leaving only a description of the collection in terms of ‘average’ or ‘bulk’ quantities: number of particles per unit volume, density of energy, density of momentum, pressure, temperature, etc. The behavior of a lake of water, and the gravitational field it generates, does not depend upon where any one particular water molecule happens to be: it depends only on the average properties of huge collections of molecules.

Nevertheless, these properties can vary from point to point in the lake: the pressure is larger at the bottom than at the top, and the temperature may vary as well. The atmosphere, another fluid, has a density that varies with position. This raises the question of how large a collection of particles to average over: it must clearly be large enough so that the individual particles don’t matter, but it must be small enough so that it is relatively homogeneous: the average velocity, kinetic energy, and interparticle spacing must be the same everywhere in the collection. Such a collection is called an ‘element’. This is a somewhat imprecise but useful term for a large collection of particles that may be regarded as having a single value for such quantities as density, average velocity, and temperature. If such a collection doesn’t exist (e.g. a very rarified gas), then the continuum approximation breaks down.

The continuum approximation assigns to each element a value of density, temperature, etc. Since the elements are regarded as ‘small’, this approximation is expressed mathematically by assigning to each *point* a value of density, temperature, etc. So a continuum is defined by various fields, having values at each point and at each time.

So far, this notion of a continuum embraces rocks as well as gases. A *fluid* is a continuum that ‘flows’: this definition is not very precise, and so the division between solids and fluids is not very well defined. Most solids will flow under high enough pressure. What makes a substance rigid? After some thought we should be able to see that rigidity comes from forces *parallel* to the interface between two elements. Two adjacent elements can push and pull on each other, but the continuum won’t be rigid unless they can also prevent each other from sliding along their common boundary. A *fluid* is characterized by the weakness of such antislapping forces compared to the direct push–pull force, which is called pressure.
A *perfect* fluid is defined as one in which *all* antislapping forces are zero, and the only force between neighboring fluid elements is pressure. We will soon see how to make this mathematically precise.

### 4.2 Dust: the number-flux vector \( \vec{N} \)

We will introduce the relativistic description of a fluid with the simplest one: ‘dust’ is defined to be a collection of particles, all of which are at rest in some one Lorentz frame. It isn’t very clear how this usage of the term ‘dust’ evolved from the other meaning as that substance which is at rest on the windowsill, but it has become a standard usage in relativity.

#### The number density \( n \)

The simplest question we can ask about these particles is: How many are there per unit volume? In their rest frame, this is merely an exercise in counting the particles and dividing by the volume they occupy. By doing this in many small regions we could come up with different numbers at different points, since the particles may be distributed more densely in one area than in another. We define this *number density* to be \( n \):

\[
\begin{align*}
n &:= \text{number density in the MCRF of the element.} \\
\end{align*}
\]

What is the number density in a frame \( \vec{O} \) in which the particles are not at rest? They will all have the same velocity \( v \) in \( \vec{O} \). If we look at the same particles as we counted up in the rest frame, then there are clearly the same *number* of particles, but they do not occupy the same volume. Suppose they were originally in a rectangular solid of dimension \( \Delta x \Delta y \Delta z \). The Lorentz contraction will reduce this to \( \Delta x \Delta y \Delta z \sqrt{1 - v^2} \), since lengths in the direction of motion contract but lengths perpendicular do not (Fig. 4.1). Because of this, the number of particles per unit volume is \( \sqrt{1 - v^2} \) times what it was in the rest frame:

\[
\frac{n}{\sqrt{1 - v^2}} = \left\{ \text{number density in frame in which particles have velocity } v \right\}.
\]

#### The flux across a surface

When particles move, another question of interest is, ‘how many’ of them are moving in a certain direction? This is made precise by the definition of flux: *the flux of particles across a surface is the number crossing a unit area of that surface in a unit time*. This clearly depends on the inertial reference frame (‘area’ and ‘time’ are frame-dependent concepts) and on the orientation of the surface (a surface parallel to the velocity of the particles
The Lorentz contraction causes the density of particles to depend upon the frame in which it is measured.

Simple illustration of the transformation of flux: if particles move only in the $x$-direction, then all those within a distance $v\Delta t$ of the surface $S$ will cross $S$ in the time $\Delta t$.

won’t be crossed by any of them). In the rest frame of the dust the flux is zero, since all particles are at rest. In the frame $\vec{O}$, suppose the particles all move with velocity $v$ in the $\vec{x}$ direction, and let us for simplicity consider a surface $S$ perpendicular to $\vec{x}$ (Fig. 4.2). The rectangular volume outlined by a dashed line clearly contains all and only those particles that will cross the area $\Delta A$ of $S$ in the time $\Delta \tilde{t}$. It has volume $v\Delta \tilde{t} \Delta A$, and contains $[n/\sqrt{(1 - v^2)}]v\Delta \tilde{t} \Delta A$ particles, since in this frame the number density is $n/\sqrt{(1 - v^2)}$. The number crossing per unit time and per unit area is the flux across surfaces of constant $\vec{x}$:

$$ (\text{flux})_{\vec{x}} = \frac{nv}{\sqrt{(1 - v^2)}}. $$

Suppose, more generally, that the particles had a $y$ component of velocity in $\vec{O}$ as well. Then the dashed line in Fig. 4.3 encloses all and only those particles that cross $\Delta A$ in $S$ in
4.2 Dust: the number–flux vector $\vec{N}$

Consider the vector $\vec{N}$ defined by

$$\vec{N} = n \vec{U}, \quad (4.4)$$

where $\vec{U}$ is the four-velocity of the particles. In a frame $\bar{O}$ in which the particles have a velocity $(v^x, v^y, v^z)$, we have

$$\vec{U} \rightarrow \bar{O} \left( \frac{1}{\sqrt{1 - v^2}}, \frac{v^x}{\sqrt{(1 - v^2)}}, \frac{v^y}{\sqrt{(1 - v^2)}}, \frac{v^z}{\sqrt{(1 - v^2)}} \right).$$

It follows that

$$\vec{N} \rightarrow \bar{O} \left( \frac{n}{\sqrt{(1 - v^2)}}, \frac{nv^x}{\sqrt{(1 - v^2)}}, \frac{nv^y}{\sqrt{(1 - v^2)}}, \frac{nv^z}{\sqrt{(1 - v^2)}} \right). \quad (4.5)$$

Thus, in any frame, the time component of $\vec{N}$ is the number density and the spatial components are the fluxes across surfaces of the various coordinates. This is a very important conceptual result. In Galilean physics, number density was a scalar, the same in all frames (no Lorentz contraction), while flux was quite another thing: a three-vector that was frame dependent, since the velocities of particles are a frame-dependent notion. Our relativistic approach has unified these two notions into a single, frame-independent four-vector. This
is progress in our thinking, of the most fundamental sort: the union of apparently disparate notions into a single coherent one.

It is worth reemphasizing the sense in which we use the word ‘frame-independent’. The vector $\vec{N}$ is a geometrical object whose existence is independent of any frame; as a tensor, its action on a one-form to give a number is independent of any frame. Its components do of course depend on the frame. Since prerelativity physicists regarded the flux as a three-vector, they had to settle for it as a frame-dependent vector, in the following sense. As a three-vector it was independent of the orientation of the spatial axes in the same sense that four-vectors are independent of all frames; but the flux three-vector is different in frames that move relative to one another, since the velocity of the particles is different in different frames. To the old physicists, a flux vector had to be defined relative to some inertial frame. To a relativist, there is only one four-vector, and the frame dependence of the older way of looking at things came from concentrating only on a set of three of the four components of $\vec{N}$. This unification of the Galilean frame-independent number density and frame-dependent flux into a single frame-independent four-vector $\vec{N}$ is similar to the unification of ‘energy’ and ‘momentum’ into four-momentum.

One final note: it is clear that

$$\vec{N} \cdot \vec{N} = -n^2, \quad n = (\vec{N} \cdot \vec{N})^{1/2}. \tag{4.6}$$

Thus, $n$ is a scalar. In the same way that ‘rest mass’ is a scalar, even though energy and ‘inertial mass’ are frame dependent, here we have that $n$ is a scalar, the ‘rest density’, even though number density is frame dependent. We will always define $n$ to be a scalar number equal to the number density in the MCRF. We will make similar definitions for pressure, temperature, and other quantities characteristic of the fluid. These will be discussed later.

## 4.3 One-forms and surfaces

### Number density as a timelike flux

We can complete the above discussion of the unity of number density and flux by realizing that number density can be regarded as a timelike flux. To see this, let us look at the flux across $x$ surfaces again, this time in a spacetime diagram, in which we plot only $\bar{t}$ and $\bar{x}$ (Fig. 4.4). The surface $S$ perpendicular to $\bar{x}$ has the world line shown. At any time $\bar{t}$ it is just one point, since we are suppressing both $\bar{y}$ and $\bar{z}$. The world lines of those particles that go through $S$ in the time $\Delta \bar{t}$ are also shown. The flux is the number of world lines that cross $S$ in the interval $\Delta \bar{t} = 1$. Really, since it is a two-dimensional surface, its ‘world path’ is three-dimensional, of which we have drawn only a section. The flux is the number of world lines that cross a unit ‘volume’ of this three-surface: by volume we of course
mean a cube of unit side, $\Delta \tilde{t} = 1, \Delta \tilde{y} = 1, \Delta \tilde{z} = 1$. So we can define a flux as the number of world lines crossing a unit three-volume. There is no reason we cannot now define this three-volume to be an ordinary spatial volume $\Delta \tilde{x} = 1, \Delta \tilde{y} = 1, \Delta \tilde{z} = 1$, taken at some particular time $\tilde{t}$. This is shown in Fig. 4.5. Now the flux is the number crossing in the interval $\Delta \tilde{x} = 1$ (since $\tilde{y}$ and $\tilde{z}$ are suppressed). But this is just the number ‘contained’ in the unit volume at the given time: the number density. So the ‘timelike’ flux is the number density.

**A one-form defines a surface**

The way we described surfaces above was somewhat clumsy. To push our invariant picture further we need a somewhat more satisfactory mathematical representation of the surface
that these world lines are crossing. This representation is given by one-forms. In general, a
surface is defined as the solution to some equation
\[ \phi(t, x, y, z) = \text{const}. \]
The gradient of the function \( \phi, \tilde{d}\phi \), is a normal one-form. In some sense, \( \tilde{d}\phi \) defines the
surface \( \phi = \text{const.} \), since it uniquely determines the directions normal to that surface. However, any multiple of \( \tilde{d}\phi \) also defines the same surface, so it is customary to use the
unit-normal one-form when the surface is not null:
\[ \tilde{n} := \tilde{d}\phi / |\tilde{d}\phi|, \]
where
\[ |\tilde{d}\phi| \text{ is the magnitude of } \tilde{d}\phi : \]
\[ |\tilde{d}\phi| = |\eta^{\alpha\beta} \phi_\alpha \phi_\beta|^{1/2}. \]
(Do not confuse \( \tilde{n} \) with \( n \), the number density in the MCRF: they are completely different,
given, by historical accident, the same letter.)

As in three-dimensional vector calculus (e.g. Gauss’ law), we define the ‘surface ele-
ment’ as the unit normal times an area element in the surface. In this case, a volume element
in a three-space whose coordinates are \( x^\alpha, x^\beta, \) and \( x^\gamma \) (for some particular values of \( \alpha, \beta, \)
and \( \gamma \), all distinct) can be represented by
\[ \tilde{n} \, dx^\alpha \, dx^\beta \, dx^\gamma, \]
and a unit volume (\( dx^\alpha = dx^\beta = dx^\gamma = 1 \)) is just \( \tilde{n} \). (These \( dx \)s are the infinitesimals that
we integrate over, not the gradients.)

The flux across the surface

Recall from Gauss’ law in three dimensions that the flux across a surface of, say, the electric
field is just \( E \cdot n \), the dot product of \( E \) with the unit normal. The situation here is exactly
the same: the flux (of particles) across a surface of constant \( \phi \) is \( \langle \tilde{n}, \tilde{N} \rangle \). To see this, let \( \phi \)
be a coordinate, say \( \bar{x} \). Then a surface of constant \( \bar{x} \) has normal \( \tilde{d}\bar{x} \), which is a unit normal
already since \( \tilde{d}\bar{x} \rightarrow \mathcal{O} (0, 1, 0, 0) \). Then \( \langle \tilde{d}\bar{x}, \tilde{N} \rangle = N^\alpha (\tilde{d}\bar{x})_\alpha = N^{\bar{x}} \), which is what we have
already seen is the flux across \( \bar{x} \) surfaces. Clearly, had we chosen \( \phi = \bar{t} \), then we would
have wound up with \( N^\bar{t} \), the number density, or flux across a surface of constant \( \bar{t} \).

This is one of the first concrete physical examples of our definition of a vector as a
function of one-forms into real numbers. Given the vector \( \tilde{N} \), we can calculate the flux
across a surface by finding the unit-normal one-form for that surface, and contracting it
with \( \tilde{N} \). We have, moreover, expressed everything frame invariantly and in a manner that
separates the property of the system of particles \( \tilde{N} \) from the property of the surface \( \tilde{n} \). All
of this will have many parallels in § 4.4 below.
Representation of a frame by a one-form

Before going on to discuss other properties of fluids, we should mention a useful fact. An inertial frame, which up to now has been defined by its four-velocity, can be defined also by a one-form, namely that associated with its four-velocity $\mathbf{g}(\vec{U}, )$. This has components

$$U_\alpha = \eta_{\alpha\beta} U^\beta$$

or, in this frame,

$$U_0 = -1, U_i = 0.$$

This is clearly also equal to $-\tilde{d}t$ (since their components are equal). So we could equally well define a frame by giving $\tilde{d}t$. This has a nice picture: $\tilde{d}t$ is to be pictured as a set of surfaces of constant $t$, the surfaces of simultaneity. These clearly do define the frame, up to spatial rotations, which we usually ignore. In fact, in some sense $\tilde{d}t$ is a more natural way to define the frame than $\vec{U}$. For instance, the energy of a particle whose four-momentum is $\vec{p}$ is

$$E = \langle \tilde{d}t, \vec{p} \rangle = p^0. \quad (4.10)$$

There is none of the awkward minus sign that we get in Eq. (2.35)

$$E = -\vec{p} \cdot \vec{U}. \quad (4.10)$$

4.4 Dust again: the stress–energy tensor

So far we have only discussed how many dust particles there are. But they also have energy and momentum, and it will turn out that their energy and momentum are the source of the gravitational field in GR. So we must now ask how to represent them in a frame-invariant manner. We will assume for simplicity that all the dust particles have the same rest mass $m$.

Energy density

In the MCRF, the energy of each particle is just $m$, and the number per unit volume is $n$. Therefore the energy per unit volume is $mn$. We denote this in general by $\rho$:

$$\rho := \text{energy density in the MCRF.} \quad (4.11)$$

Thus $\rho$ is a scalar just as $n$ is (and $m$ is). In our case of dust,

$$\rho = nm \text{ (dust).} \quad (4.12)$$

In more general fluids, where there is random motion of particles and hence kinetic energy of motion, even in an average rest frame, Eq. (4.12) will not be valid.
In the frame $\bar{\mathcal{O}}$ we again have that the number density is $n/\sqrt{1-v^2}$, but now the energy of each particle is $m/\sqrt{1-v^2}$, since it is moving. Therefore the energy density is $\frac{mn}{\sqrt{1-v^2}}$:

$$\rho = \left\{ \begin{array}{l} \text{energy density in a frame in which particles have velocity } v \\ \end{array} \right\}. \quad (4.13)$$

This transformation involves two factors of $(1-v^2)^{-1/2} = \Lambda^{\bar{\alpha} \bar{0}}$, because both volume and energy transform. It is impossible, therefore, to represent energy density as some component of a vector. It is, in fact, a component of a $(\bar{\alpha} \bar{\beta})$ tensor. This is most easily seen from the point of view of our definition of a tensor. To define energy requires a one-form, in order to select the zero component of the four-vector of energy and momentum; to define a density also requires a one-form, since density is a flux across a constant-time surface. Similarly, an energy flux also requires two one-forms: one to define ‘energy’ and the other to define the surface. We can also speak of momentum density: again a one-form defines which component of momentum, and another one-form defines density. By analogy there is also momentum flux: the rate at which momentum crosses some surface. All these things require two one-forms. So there is a tensor $T$, called the stress–energy tensor, which has all these numbers as values when supplied with the appropriate one-forms as arguments.

**Stress–energy tensor**

The most convenient definition of the stress–energy tensor is in terms of its components in some (arbitrary) frame:

$$T(\dd x^\alpha, \dd x^\beta) = T_{\alpha\beta} := \left\{ \begin{array}{l} \text{flux of } \alpha \text{ momentum across } \alpha \text{ surface} \\ \end{array} \right\}. \quad (4.14)$$

(By $\alpha$ momentum we mean, of course, the $\alpha$ component of four-momentum: $p^\alpha := \langle \dd x^\alpha, \vec{p} \rangle$.) That this is truly a tensor is proved in Exer. 5, § 4.10.

Let us see how this definition fits in with our discussion above. Consider $T^{00}$. This is defined as the flux of zero momentum (energy) across a surface $t = \text{constant}$. This is just the energy density:

$$T^{00} = \text{energy density.} \quad (4.15)$$

Similarly, $T^{0i}$ is the flux of energy across a surface $x^i = \text{const}$:

$$T^{0i} = \text{energy flux across } x^i \text{ surface.} \quad (4.16)$$

Then $T^{i0}$ is the flux of $i$ momentum across a surface $t = \text{const}$: the density of $i$ momentum,

$$T^{i0} = i \text{ momentum density.} \quad (4.17)$$

Finally, $T^{ij}$ is the $j$ flux of $i$ momentum:

$$T^{ij} = \text{flux of } i \text{ momentum across } j \text{ surface.} \quad (4.18)$$
For any particular system, giving the components of $T$ in some frame defines it completely. For dust, the components of $T$ in the MCRF are particularly easy. There is no motion of the particles, so all $i$ momenta are zero and all spatial fluxes are zero. Therefore

$$(T^{00})_{\text{MCRF}} = \rho = mn,$$

$$(T^{0i})_{\text{MCRF}} = (T^{i0})_{\text{MCRF}} = (T^{ij})_{\text{MCRF}} = 0.$$  

It is easy to see that the tensor $\tilde{p} \otimes \tilde{N}$ has exactly these components in the MCRF, where $\tilde{p} = m\tilde{U}$ is the four-momentum of a particle. Therefore we have

$$\text{Dust : } T = \tilde{p} \otimes \tilde{N} = mn \tilde{U} \otimes \tilde{U} = \rho \tilde{U} \otimes \tilde{U}. \quad (4.19)$$

From this we can conclude

$$T^{\alpha\beta} = T(\tilde{\omega}^{\alpha}, \tilde{\omega}^{\beta})$$
$$= \rho \tilde{U}(\tilde{\omega}^{\alpha})\tilde{U}(\tilde{\omega}^{\beta})$$
$$= \rho \tilde{U}^{\alpha} \tilde{U}^{\beta}. \quad (4.20)$$

In the frame $\tilde{O}$, where

$$\tilde{U} \rightarrow \left( \frac{1}{\sqrt{(1 - v^2)}}, \frac{v^x}{\sqrt{(1 - v^2)}}, \ldots \right),$$

we therefore have

$$\begin{cases}
T^{00} = \rho \tilde{U}^{0} \tilde{U}^{0} = \rho/(1 - v^2), \\
T^{0i} = \rho \tilde{U}^{0} \tilde{U}^{i} = \rho v^i/(1 - v^2), \\
T^{i0} = \rho \tilde{U}^{i} \tilde{U}^{0} = \rho v^i(1 - v^2), \\
T^{ij} = \rho \tilde{U}^{i} \tilde{U}^{j} = \rho v^i v^j/(1 - v^2).
\end{cases} \quad (4.21)$$

These are exactly what we would calculate, from first principles, for energy density, energy flux, momentum density, and momentum flux respectively. (We did the calculation for energy density above.) Notice one important point: $T^{\alpha\beta} = T^{\beta\alpha}$; that is, $T$ is symmetric. This will turn out to be true in general, not just for dust.

## 4.5 General fluids

Until now we have dealt with the simplest possible collection of particles. To generalize this to real fluids, we have to take account of the facts that (i) besides the bulk motions of the fluid, each particle has some random velocity; and (ii) there may be various forces between particles that contribute potential energies to the total.
**Table 4.1** Macroscopic quantities for single-component fluids

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{U}$</td>
<td>Four-velocity of fluid element</td>
<td>Four-velocity of MCRF</td>
</tr>
<tr>
<td>$n$</td>
<td>Number density</td>
<td>Number of particles per unit volume in MCRF</td>
</tr>
<tr>
<td>$\vec{N}$</td>
<td>Flux vector</td>
<td>$\vec{N} := n\vec{U}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Energy density</td>
<td>Density of total mass energy (rest mass, random kinetic, chemical, ...).</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>Internal energy per particle</td>
<td>$\Pi := (\rho/n) - m \Rightarrow \rho = n(m + \Pi)$ Thus $\Pi$ is a general name for all energies other than the rest mass.</td>
</tr>
<tr>
<td>$\rho_0$</td>
<td>Rest-mass density</td>
<td>$\rho_0 := mn$. Since $m$ is a constant, this is the ‘energy’ associated with the rest mass only. Thus, $\rho = \rho_0 + n\Pi$.</td>
</tr>
<tr>
<td>$T$</td>
<td>Temperature</td>
<td>Usual thermodynamic definition in MCRF (see below).</td>
</tr>
<tr>
<td>$p$</td>
<td>Pressure</td>
<td>Usual fluid-dynamical notion in MCRF. More about this later.</td>
</tr>
<tr>
<td>$S$</td>
<td>Specific entropy</td>
<td>Entropy per particle (see below).</td>
</tr>
</tbody>
</table>

**Definition of macroscopic quantities**

The concept of a fluid element was discussed in § 4.1. For each fluid element, we go to the frame in which it is at rest (its total spatial momentum is zero). This is its MCRF. This frame is truly *momentarily* comoving: since fluid elements can be accelerated, a moment later a different inertial frame will be the MCRF. Moreover, two different fluid elements may be moving relative to one another, so that they would not have the same MCRFs. Thus, the MCRF is specific to a single fluid element, and which frame is the MCRF is a function of position and time. *All scalar quantities associated with a fluid element in relativity* (such as number density, energy density, and temperature) *are defined to be their values in the MCRF*. Thus we make the definitions displayed in Table 4.1. We confine our attention to fluids that consist of only one component, one kind of particle, so that (for example) interpenetrating flows are not possible.

**First law of thermodynamics**

This law is simply a statement of conservation of energy. In the MCRF, we imagine that the fluid element is able to exchange energy with its surroundings in only two ways: by heat conduction (absorbing an amount of heat $\Delta Q$) and by work (doing an amount of work $p\Delta V$, where $V$ is the three-volume of the element). If we let $E$ be the total energy of the
element, then since $\Delta Q$ is energy gained and $p\Delta V$ is energy lost, we can write (assuming small changes)

$$\begin{align*}
\Delta E &= \Delta Q - p\Delta V, \\
\text{or} \\
\Delta Q &= \Delta E + p\Delta V.
\end{align*}$$

(4.22)

Now, if the element contains a total of $N$ particles, and if this number doesn’t change (i.e. no creation or destruction of particles), we can write

$$V = \frac{N}{n}, \quad \Delta V = -\frac{N}{n^2}\Delta n.$$  

(4.23)

Moreover, we also have (from the definition of $\rho$)

$$E = \rho V = \rho N/n, \quad \Delta E = \rho \Delta V + V \Delta \rho.$$  

These two results imply

$$\Delta Q = \frac{N}{n} \Delta \rho - N(\rho + p)\frac{\Delta n}{n^2}.$$  

If we write $q := Q/N$, which is the heat absorbed per particle, we obtain

$$n \Delta q = \Delta \rho - \frac{\rho + p}{n} \Delta n.$$  

(4.24)

Now suppose that the changes are ‘infinitesimal’. It can be shown in general that a fluid’s state can be given by two parameters: for instance, $\rho$ and $T$ or $\rho$ and $n$. Everything else is a function of, say, $\rho$ and $n$. That means that the right-hand side of Eq. (4.24),

$$d\rho - (\rho + p)dn/n,$$

depends only on $\rho$ and $n$. The general theory of first-order differential equations shows that this always possesses an integrating factor: that is, there exist two functions $A$ and $B$, functions only of $\rho$ and $n$, such that

$$d\rho - (\rho + p)dn/n \equiv A dB$$

is an identity for all $\rho$ and $n$. It is customary in thermodynamics to define temperature to be $A/n$ and specific entropy to be $B$:

$$d\rho - (\rho + p)dn/n = nT dS,$$  

(4.25)

or, in other words,

$$\Delta q = T \Delta S.$$  

(4.26)

The heat absorbed by a fluid element is proportional to its increase in entropy.

We have thus introduced $T$ and $S$ as convenient mathematical definitions. A full treatment would show that $T$ is the thing normally meant by temperature, and that $S$ is the thing
used in the second law of thermodynamics, which says that the total entropy in any system must increase. We’ll have nothing to say about the second law. Entropy appears here only because it is an integral of the first law, which is merely conservation of energy. In particular, we shall use both Eqs. (4.25) and (4.26) later.

The general stress–energy tensor

The definition of $T^{\alpha\beta}$ in Eq. (4.14) is perfectly general. Let us in particular look at it in the MCRF, where there is no bulk flow of the fluid element, and no spatial momentum in the particles. Then in the MCRF we have:

1. $T^{00} = \text{energy density} = \rho$.
2. $T^{0i} = \text{energy flux. Although there is no motion in the MCRF, energy may be transmitted by heat conduction. So } T^{0i} \text{ is basically a heat-conduction term in the MCRF.}$
3. $T^{ij} = \text{momentum density. Again the particles themselves have no net momentum in the MCRF, but if heat is being conducted, then the moving energy will have an associated momentum. We’ll argue below that } T^{00} \equiv T^{0i}$.
4. $T^{ij} = \text{momentum flux. This is an interesting and important term. The next section gives a thorough discussion of it. It is called the stress.}$

The spatial components of $T$, $T^{ij}$

By definition, $T^{ij}$ is the flux of $i$ momentum across the $j$ surface. Consider (Fig. 4.6) two adjacent fluid elements, represented as cubes, having the common interface $S$. In general, they exert forces on each other. Shown in the diagram is the force $F$ exerted by $A$ on $B$ ($B$ of course exerts an equal and opposite force on $A$). Since force equals the rate of change of momentum (by Newton’s law, which is valid here, since we are in the MCRF where

![Figure 4.6](image-url)

The force $F$ exerted by element $A$ on its neighbor $B$ may be in any direction depending on properties of the medium and any external forces.
velocities are zero), \( A \) is pouring momentum into \( B \) at the rate \( \mathbf{F} \) per unit time. Of course, \( B \) may or may not acquire a new velocity as a result of this new momentum it acquires; this depends upon how much momentum is put into \( B \) by its other neighbors. Obviously \( B \)'s motion is the resultant of all the forces. Nevertheless, each force adds momentum to \( B \). There is therefore a flow of momentum across \( S \) from \( A \) to \( B \) at the rate \( \mathbf{F} \). If \( S \) has area \( A \), then the flux of momentum across \( S \) is \( \mathbf{F}/A \). If \( S \) is a surface of constant \( x^j \), then \( T^{ij} \) for fluid element \( A \) is \( F^i/A \).

This is a brief illustration of the meaning of \( T^{ij} \): it represents forces between adjacent fluid elements. As mentioned before, these forces need not be perpendicular to the surfaces between the elements (i.e. viscosity or other kinds of rigidity give forces parallel to the interface). But if the forces are perpendicular to the interfaces, then \( T^{ij} \) will be zero unless \( i = j \). (Think this through – we’ll use it shortly.)

**Symmetry of \( T^{\alpha\beta} \) in MCRF**

We now prove that \( T \) is a symmetric tensor. We need only prove that its components are symmetric in one frame; that implies that for any \( \tilde{r}, \tilde{q} \), \( T(\tilde{r}, \tilde{q}) = T(\tilde{q}, \tilde{r}) \), which implies the symmetry of its components in any other frame. The easiest frame is the MCRF.

(a) Symmetry of \( T^{ij} \). Consider Fig. 4.7 in which we have drawn a fluid element as a cube of side \( l \). The force it exerts on a neighbor across surface (1) (a surface \( x = \text{const.} \)) is \( F^i_1 = T^{ij}l^2 \), where the factor \( l^2 \) gives the area of the face. Here, \( i \) runs over 1, 2, and 3, since \( \mathbf{F} \) is not necessarily perpendicular to the surface. Similarly, the force it exerts on a neighbor across (2) is \( F^j_2 = T^{ji}l^2 \). (We shall take the limit \( l \to 0 \), so bear in mind that the element is small.) The element also exerts a force on its neighbor toward the \(-x\) direction, which we call \( F^i_j \). Similarly, there is \( F^i_4 \) on the face looking in the negative \( y \) direction. The forces on the fluid element are, respectively, \(-F^i_1, -F^i_2, \) etc. The first point is that \( F^i_3 \approx -F^i_4 \) in order that the sum of the forces on the element should vanish when \( l \to 0 \) (otherwise the tiny mass obtained as \( l \to 0 \) would have an infinite acceleration). The next point is to compute

![Figure 4.7](image_url)
torques about the $z$ axis through the center of the fluid element. (Since forces on the top and bottom of the cube don’t contribute to this, we haven’t considered them.) For the torque calculation it is convenient to place the origin of coordinates at the center of the cube. The torque due to $-F_1$ is $-(r \times F_1) \cdot e_z = -xF_1^y = -\frac{1}{2}IT^{xy}l^2$, where we have approximated the force as acting at the center of the face, where $r \to (l/2, 0, 0)$ (note particularly that $y = 0$ there). The torque due to $-F_3$ is the same, $-\frac{1}{2}lT^{yx}$. The torque due to $-F_2$ is $-(r \times F_2) \cdot e_z = +yF_2^x = \frac{1}{2}IT^{xy}l^2$. Similarly, the torque due to $-F_4$ is the same, $\frac{1}{2}lT^{xy}$. Therefore, the total torque is

$$\tau_z = \frac{l^3}{2}(T^{xy} - T^{yx}). \quad (4.27)$$

The moment of inertia of the element about the $z$ axis is proportional to its mass times $l^2$, or

$$I = \alpha \rho l^5,$$

where $\alpha$ is some numerical constant and $\rho$ is the density (whether of total energy or rest mass doesn’t matter in this argument). Therefore the angular acceleration is

$$\dot{\theta} = \frac{\tau}{I} = \frac{T^{xy} - T^{yx}}{\alpha \rho l^2}. \quad (4.28)$$

Since $\alpha$ is a number and $\rho$ is independent of the size of the element, as are $T^{xy}$ and $T^{yx}$, this will go to infinity as $l \to 0$ unless

$$T^{xy} = T^{yx}.$$

Thus, since it is obviously not true that fluid elements are whirling around inside fluids, smaller ones whirling ever faster, we have that the stresses are always symmetric:

$$T^{ij} = T^{ji}. \quad (4.29)$$

(b) Equality of momentum density and energy flux. This is much easier to demonstrate. The energy flux is the density of energy times the speed it flows at. But since energy and mass are the same, this is the density of mass times the speed it is moving at; in other words, the density of momentum. Therefore $T^{0i} = T^{ri}$.

### Conservation of energy–momentum

Since $T$ represents the energy and momentum content of the fluid, there must be some way of using it to express the law of conservation of energy and momentum. In fact it is reasonably easy. In Fig. 4.8 we see a cubical fluid element, seen only in cross-section ($z$ direction suppressed). Energy can flow in across all sides. The rate of flow across face (4) is $l^2T^{0x}(x = 0)$, and across (2) is $-l^2T^{0x}(x = a)$; the second term has a minus sign, since $T^{0x}$ represents energy flowing in the positive $x$ direction, which is out of the volume across
Figure 4.8

A section \( z = \text{const.} \) of a cubical fluid element.

face (2). Similarly, energy flowing in the \( y \) direction is \( l^2 T^{0y}(y = 0) - l^2 T^{0y}(y = l) \). The sum of these rates must be the rate of increase in the energy inside, \( \partial(T^{00}l^3)/\partial t \) (statement of conservation of energy). Therefore we have

\[
\frac{\partial}{\partial t} l^3 T^{00} = l^2 \left[ T^{0x}(x = 0) - T^{0x}(x = l) + T^{0y}(y = 0) - T^{0y}(y = l) \right].
\] (4.30)

Dividing by \( l^3 \) and taking the limit \( l \to 0 \) gives

\[
\frac{\partial}{\partial t} T^{00} = -\frac{\partial}{\partial x} T^{0x} - \frac{\partial}{\partial y} T^{0y} - \frac{\partial}{\partial z} T^{0z}.
\] (4.31)

[In deriving this we use the definition of the derivative

\[
\lim_{l \to 0} \frac{T^{0x}(x = 0) - T^{0x}(x = l)}{l} = -\frac{\partial}{\partial x} T^{0x}.
\] (4.32)

Eq. (4.31) can be written as

\[
T^{00},0 + T^{0x},x + T^{0y},y + T^{0z},z = 0
\]

or

\[
T^{0\alpha},\alpha = 0.
\] (4.33)

This is the statement of the law of conservation of energy.

Similarly, momentum is conserved. The same mathematics applies, with the index ‘0’ changed to whatever spatial index corresponds to the component of momentum whose conservation is being considered. The general conservation law is, then,

\[
T^{\alpha\beta},\beta = 0.
\] (4.34)

This applies to any material in SR. Notice it is just a four-dimensional divergence. Its relation to Gauss’ theorem, which gives an integral form of the conservation law, will be discussed later.
Conservation of particles

It may also happen that, during any flow of the fluid, the number of particles in a fluid element will change, but of course the total number of particles in the fluid will not change. In particular, in Fig. 4.8 the rate of change of the number of particles in a fluid element will be due only to loss or gain across the boundaries, i.e. to net fluxes out or in. This conservation law is derivable in the same way as Eq. (4.34) was. We can then write that

$$\frac{\partial}{\partial t} N^0 = - \frac{\partial}{\partial x} N_x - \frac{\partial}{\partial y} N_y - \frac{\partial}{\partial z} N_z$$

or

$$N^\alpha, _{\alpha} = (nU^\alpha)_{, \alpha} = 0. \quad (4.35)$$

We will confine ourselves to discussing only fluids that obey this conservation law. This is hardly any restriction, since $n$ can, if necessary, always be taken to be the density of baryons.

‘Baryon’, for those not familiar with high-energy physics, is a general name applied to the more massive particles in physics. The two commonest are the neutron and proton. All others are too unstable to be important in everyday physics – but when they decay they form protons and neutrons, thus conserving the total number of baryons without conserving rest mass or particle identity. Although theoretical physics suggests that baryons may not always be conserved – for instance, so-called ‘grand unified theories’ of the strong, weak, and electromagnetic interactions may predict a finite lifetime for the proton, and collapse to and subsequent evaporation of a black hole (see Ch. 11) will not conserve baryon number – no such phenomena have yet been observed and, in any case, are unlikely to be important in most situations.

4.6 Perfect fluids

Finally, we come to the type of fluid which is our principal subject of interest. A perfect fluid in relativity is defined as a fluid that has no viscosity and no heat conduction in the MCRF. It is a generalization of the ‘ideal gas’ of ordinary thermodynamics. It is, next to dust, the simplest kind of fluid to deal with. The two restrictions in its definition simplify enormously the stress–energy tensor, as we now see.

No heat conduction

From the definition of $T$, we see that this immediately implies that, in the MCRF, $T^{0i} = T^{i0} = 0$. Energy can flow only if particles flow. Recall that in our discussion of the first law of thermodynamics we showed that if the number of particles was conserved, then
the specific entropy was related to heat flow by Eq. (4.26). This means that in a perfect fluid, if Eq. (4.35) for conservation of particles is obeyed, then we should also have that $S$ is a constant in time during the flow of the fluid. We shall see how this comes out of the conservation laws in a moment.

**No viscosity**

Viscosity is a force parallel to the interface between particles. Its absence means that the forces should always be perpendicular to the interface, i.e. that $T^{ij}$ should be zero unless $i = j$. This means that $T^{ij}$ should be a diagonal matrix. Moreover, it must be diagonal in all MCRF frames, since ‘no viscosity’ is a statement independent of the spatial axes. The only matrix diagonal in all frames is a multiple of the identity: all its diagonal terms are equal. Thus, an $x$ surface will have across it only a force in the $x$ direction, and similarly for $y$ and $z$; these forces-per-unit-area are all equal, and are called the pressure, $p$. So we have $T^{ij} = p\delta^{ij}$. From six possible quantities (the number of independent elements in the $3 \times 3$ symmetric matrix $T^{ij}$) the zero-viscosity assumption has reduced the number of functions to one, the pressure.

**Form of $T$**

In the MCRF, $T$ has the components we have just deduced:

$$
(T^\alpha{}^\beta) = \begin{pmatrix}
\rho & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & p & 0 \\
0 & 0 & 0 & p
\end{pmatrix}.
$$

(4.36)

It is not hard to show that in the MCRF

$$
T^\alpha{}^\beta = (\rho + p)U^\alpha U^\beta + p\eta^\alpha{}^\beta.
$$

(4.37)

For instance, if $\alpha = \beta = 0$, then $U^0 = 1, \eta^{00} = -1$, and $T^{00} = (\rho + p) - p = \rho$, as in Eq. (4.36). By trying all possible $\alpha$ and $\beta$ you can verify that Eq. (4.37) gives Eq. (4.36). But Eq. (4.37) is a frame-invariant formula in the sense that it uniquely implies

$$
T = (\rho + p)\tilde{U} \otimes \tilde{U} + pg^{-1}.
$$

(4.38)

This is the stress–energy tensor of a perfect fluid.

**Aside on the meaning of pressure**

A comparison of Eq. (4.38) with Eq. (4.19) shows that ‘dust’ is the special case of a pressure-free perfect fluid. This means that a perfect fluid can be pressure free only if its
particles have no random motion at all. Pressure arises in the random velocities of the particles. Even a gas so dilute as to be virtually collisionless has pressure. This is because pressure is the flux of momentum; whether this comes from forces or from particles crossing a boundary is immaterial.

The conservation laws

Eq. (4.34) gives us

\[ T^{\alpha\beta}_{\cdot\beta} = [(\rho + p)U^\alpha U^\beta + p\eta^{\alpha\beta}]_{\cdot\beta} = 0. \]  

(4.39)

This gives us our first real practice with tensor calculus. There are four equations in Eq. (4.39), one for each \( \alpha \). First, let us also assume

\[ (nU^\beta)_{\cdot\beta} = 0 \]  

(4.40)

and write the first term in Eq. (4.39) as

\[ \left[(\rho + p)U^\alpha U^\beta\right]_{\cdot\beta} = \left[\left(\frac{\rho + p}{n}\right)U^\alpha U^\beta \right]_{\cdot\beta} = nU^\beta \left(\frac{\rho + p}{n}\right)U^\alpha_{\cdot\beta}. \]  

(4.41)

Moreover, \( \eta^{\alpha\beta} \) is a constant matrix, so \( \eta^{\alpha\beta}_{\cdot\gamma} = 0 \). This also implies, by the way, that

\[ U^\alpha_{\cdot\beta} U_\alpha = 0. \]  

(4.42)

The proof of Eq. (4.42) is

\[ U^\alpha U_\alpha = -1 \Rightarrow (U^\alpha U_\alpha)_{\cdot\beta} = 0 \]  

(4.43)

or

\[ (U^\alpha U^\gamma \eta_{\alpha\gamma})_{\cdot\beta} = (U^\alpha U^\gamma)_{\cdot\beta} \eta_{\alpha\gamma} = 2U^\alpha_{\cdot\beta} U^\gamma_{\cdot\beta} \eta_{\alpha\gamma}. \]  

(4.44)

The last step follows from the symmetry of \( \eta_{\alpha\beta} \), which means that \( U^\alpha_{\cdot\beta} U^\gamma_{\cdot\beta} \eta_{\alpha\gamma} = U^\alpha U^\gamma_{\cdot\beta} \eta_{\alpha\gamma} \). Finally, the last expression in Eq. (4.44) converts to

\[ 2U^\alpha_{\cdot\beta} U_\alpha. \]

which is zero by Eq. (4.43). This proves Eq. (4.42). We can make use of Eq. (4.42) in the following way. The original equation now reads, after use of Eq. (4.41),

\[ nU^\beta \left(\frac{\rho + p}{n}\right)U^\alpha_{\cdot\beta} + p_{\cdot\beta} \eta^{\alpha\beta} U_\alpha = 0. \]  

(4.45)

From the four equations here, we can obtain a particularly useful one. Multiply by \( U_\alpha \) and sum on \( \alpha \). This gives the time component of Eq. (4.45) in the MCRF:

\[ nU^\beta U_\alpha \left(\frac{\rho + p}{n}\right)_{\cdot\beta} + p_{\cdot\beta} \eta^{\alpha\beta} U_\alpha = 0. \]  

(4.46)
The last term is just
\[ p,\beta U^\beta, \]
which we know to be the derivative of \( \rho \) along the world line of the fluid element, \( dp/d\tau \). The first term gives zero when the \( \beta \) derivative operates on \( U^\alpha \) (by Eq. (4.42)), so we obtain (using \( U^\alpha U_\alpha = -1 \))
\[ U^\beta \left[ -n \left( \frac{\rho + p}{n} \right)_{,\beta} + p,\beta \right] = 0. \quad (4.47) \]
A little algebra converts this to
\[ -U^\beta \left[ \rho,\beta - \frac{\rho + p}{n} n,\beta \right] = 0. \quad (4.48) \]
Written another way,
\[ \frac{d\rho}{d\tau} - \frac{\rho + p}{n} \frac{dn}{d\tau} = 0. \quad (4.49) \]
This is to be compared with Eq. (4.25). It means
\[ U^\alpha S,\alpha = \frac{dS}{d\tau} = 0. \quad (4.50) \]
Thus, the flow of a particle-conserving perfect fluid conserves specific entropy. This is called \textit{adiabatic}. Because entropy is constant in a fluid element as it flows, we shall not normally need to consider it. Nevertheless, it is important to remember that the law of conservation of energy in thermodynamics is embodied in the component of the conservation equations, Eq. (4.39), parallel to \( U^\alpha \).

The remaining three components of Eq. (4.39) are derivable in the following way. We write, again, Eq. (4.45):
\[ nU^\beta \left( \frac{\rho + p}{n} U^\alpha \right)_{,\beta} + p,\beta \eta^{\alpha\beta} = 0 \]
and go to the MCRF, where \( U^i = 0 \) but \( U^i,\beta \neq 0 \). In the MCRF, the zero component of this equation is the same as its contraction with \( U_\alpha \), which we have just examined. So we only need the \( i \) components:
\[ nU^\beta \left( \frac{\rho + p}{n} U^i \right)_{,\beta} + p,\beta \eta^{i\beta} = 0. \quad (4.51) \]
Since \( U^i = 0 \), the \( \beta \) derivative of \( (\rho + p)/n \) contributes nothing, and we get
\[ (\rho + p)U^{i,\beta} U^\beta = p,\beta \eta^{i\beta} = 0. \quad (4.52) \]
Lowering the index \( i \) makes this easier to read (and changes nothing). Since \( \eta^{i\beta} = \delta^{i\beta} \) we get
\[ (\rho + p)U_{i,\beta} U^\beta + p,i = 0. \quad (4.53) \]
Finally, we recall that \( U_{i\beta} U^\beta \) is the definition of the four-acceleration \( a_i \):

\[
(\rho + p)a_i + p_i = 0. \tag{4.54}
\]

Those familiar with nonrelativistic fluid dynamics will recognize this as the generalization of

\[
\rho a + \nabla p = 0, \tag{4.55}
\]

where

\[ a = \dot{v} + (v \cdot \nabla)v. \tag{4.56} \]

The only difference is the use of \((\rho + p)\) instead of \(\rho\). In relativity, \((\rho + p)\) plays the role of ‘inertial mass density’, in that, from Eq. (4.54), the larger \((\rho + p)\), the harder it is to accelerate the object. Eq. (4.54) is essentially \( F = ma \), with \(-p_i\) being the force per unit volume on a fluid element. Roughly speaking, \( p \) is the force a fluid element exerts on its neighbor, so \(-p\) is the force on the element. But the neighbor on the opposite side of the element is pushing the other way, so only if there is a change in \( p \) across the fluid element will there be a net force causing it to accelerate. That is why \(-\nabla p\) gives the force.

### 4.7 Importance for general relativity

General relativity is a relativistic theory of gravity. We weren’t able to plunge into it immediately because we lacked a good enough understanding of tensors, of fluids in SR, and of curved spaces. We have yet to study curvature (that comes next), but at this point we can look ahead and discern the vague outlines of the theory we shall study.

The first comment is on the supreme importance of \( T \) in GR. Newton’s theory has as a source of the field the density \( \rho \). This was understood to be the mass density, and so is closest to our \( \rho_0 \). But a theory that uses rest mass only as its source would be peculiar from a relativistic viewpoint, since rest mass and energy are interconvertible. In fact, we can show that such a theory would violate some very high-precision experiments (to be discussed later). So the source of the field should be all energies, the density of total mass energy \( T^{00} \). But to have as the source of the field only one component of a tensor would give a noninvariant theory of gravity: we would need to choose a preferred frame in order to calculate \( T^{00} \). Therefore Einstein guessed that the source of the field ought to be \( T \): all stresses and pressures and momenta must also act as sources. Combining this with his insight into curved spaces led him to GR.

The second comment is about pressure, which plays a more fundamental role in GR than in Newtonian theory: first, because it is a source of the field; and, second, because of its appearance in the \((\rho + p)\) term in Eq. (4.54). Consider a dense star, whose strong gravitational field requires a large pressure gradient. How large is measured by the acceleration the fluid element would have, \( a_i \), in the absence of pressure. Given the field, and
hence given $a_i$, the required pressure gradient is just that which would cause the opposite acceleration without gravity:

$$-a_i = \frac{p_i}{\rho + p}.$$  

This gives the pressure gradient $p_i$. Since $(\rho + p)$ is greater than $\rho$, the gradient must be larger in relativity than in Newtonian theory. Moreover, since all components of $\mathbf{T}$ are sources of the gravitational field, this larger pressure adds to the gravitational field, causing even larger pressures (compared to Newtonian stars) to be required to hold the star up. For stars where $p \ll \rho$ (see below), this doesn’t make much difference. But when $p$ becomes comparable to $\rho$, we find that increasing the pressure is self-defeating: no pressure gradient will hold the star up, and gravitational collapse must occur. This description, of course, glosses over much detailed calculation, but it shows that even by studying fluids in SR we can begin to appreciate some of the fundamental changes GR brings to gravitation.

Let us just remind ourselves of the relative sizes of $p$ and $\rho$. We saw in Exer. 1, § 1.14, that $p \ll \rho$ in ordinary situations. In fact, we only get $p \approx \rho$ for very dense material (neutron star) or material so hot that the particles move at close to the speed of light (a ‘relativistic’ gas).

### 4.8 Gauss’ law

Our final topic on fluids is the integral form of the conservation laws, which are expressed in differential form in Eqs. (4.34) and (4.35). As in three-dimensional vector calculus, the conversion of a volume integral of a divergence into a surface integral is called Gauss’ law. The proof of the theorem is exactly the same as in three dimensions, so we shall not derive it in detail:

$$\int V^{\alpha \alpha} \, d^4 x = \oint V^{\alpha \alpha} n_\alpha \, d^3 S, \quad (4.57)$$

where $\tilde{n}$ is the unit-normal one-form discussed in § 4.3, and $d^3 S$ denotes the three-volume of the three-dimensional hypersurface bounding the four-dimensional volume of integration. The sense of the normal is that it is outward pointing, of course, just as in three dimensions. In Fig. 4.9 a simple volume is drawn, in order to illustrate the meaning of Eq. (4.57). The volume is bounded by four pairs of hypersurfaces, for constant $t, x, y,$ and $z$; only two pairs are shown, since we can only draw two dimensions easily. The normal on the $t_2$ surface is $\tilde{d} t$. The normal on the $t_1$ surface is $-\tilde{d} t$, since ‘outward’ is clearly
backwards in time. The normal on $x_2$ is $\tilde{\mathbf{d}}x$, and on $x_1$ is $-\tilde{\mathbf{d}}x$. So the surface integral in Eq. (4.57) is

$$
\int_{t_2} V^0 \, dx \, dy \, dz + \int_{t_1} (-V^0) \, dx \, dy \, dz \\
+ \int_{x_2} V^x \, dt \, dy \, dz + \int_{x_1} (-V^x) \, dt \, dy \, dz \\
+ \text{similar terms for the other surfaces in the boundary.}
$$

We can rewrite this as

$$
\int \left[ V^0(t_2) - V^0(t_1) \right] \, dx \, dy \, dz \\
+ \int \left[ V^x(x_2) - V^x(x_1) \right] \, dt \, dy \, dz + \cdots. \tag{4.58}
$$

If we let $\tilde{V}$ be $\tilde{N}$, then $N^\alpha_{\alpha} = 0$ means that the above expression vanishes, which has the interpretation that change in the number of particles in the three-volume (first integral) is due to the flux across its boundaries (second and subsequent terms). If we are talking about energy conservation, we replace $N^\alpha$ with $T^0\alpha$, and use $T^\alpha_{\alpha,\alpha} = 0$. Then, obviously, a similar interpretation of Eq. (4.58) applies. Gauss’ law gives an integral version of energy conservation.

### 4.9 Further reading

Continuum mechanics and conservation laws are treated in most texts on GR, such as Misner et al. (1973). Students whose background in thermodynamics or fluid mechanics is weak are referred to the classic works of Fermi (1956) and Landau and Lifshitz (1959).
respectively. Apart from Exer. 25, § 4.10 below, we do not study much about electromagnetism, but it has a stress–energy tensor and illustrates conservation laws particularly clearly. See Landau and Lifshitz (1962) or Jackson (1975). Relativistic fluids with dissipation present their own difficulties, which reward close study. See Israel and Stewart (1980). Another model for continuum systems is the collisionless gas; see Andréasson (2005) for a description of how to treat such systems in GR.

### 4.10 Exercises

1 Comment on whether the continuum approximation is likely to apply to the following physical systems: (a) planetary motions in the solar system; (b) lava flow from a volcano; (c) traffic on a major road at rush hour; (d) traffic at an intersection controlled by stop signs for each incoming road; (e) plasma dynamics.

2 Flux across a surface of constant \( x \) is often loosely called ‘flux in the \( x \) direction’. Use your understanding of vectors and one-forms to argue that this is an inappropriate way of referring to a flux.

3 (a) Describe how the Galilean concept of momentum is frame dependent in a manner in which the relativistic concept is not.

   (b) How is this possible, since the relativistic definition is nearly the same as the Galilean one for small velocities? (Define a Galilean four-momentum vector.)

4 Show that the number density of dust measured by an arbitrary observer whose four-velocity is \( \vec{U}_{\text{obs}} \) is \( -\vec{N} \cdot \vec{U}_{\text{obs}} \).

5 Complete the proof that Eq. (4.14) defines a tensor by arguing that it must be linear in both its arguments.

6 Establish Eq. (4.19) from the preceding equations.

7 Derive Eq. (4.21).

8 (a) Argue that Eqs. (4.25) and (4.26) can be written as relations among one-forms, i.e.

\[
\tilde{d} \rho - (\rho + p) \tilde{d} n/n = nT \tilde{d} S = n\tilde{q}.
\]

   (b) Show that the one-form \( \tilde{q} \) is not a gradient, i.e. is not \( \tilde{d} q \) for any function \( q \).

9 Show that Eq. (4.34), when \( \alpha \) is any spatial index, is just Newton’s second law.

10 Take the limit of Eq. (4.35) for \( |v| \ll 1 \) to get

\[
\partial n/\partial t + \partial (nv^i)/\partial x^i = 0.
\]

11 (a) Show that the matrix \( \delta^{ij} \) is unchanged when transformed by a rotation of the spatial axes.

   (b) Show that any matrix which has this property is a multiple of \( \delta^{ij} \).

12 Derive Eq. (4.37) from Eq. (4.36).

13 Supply the reasoning in Eq. (4.44).

14 Argue that Eq. (4.46) is the time component of Eq. (4.45) in the MCRF.

15 Derive Eq. (4.48) from Eq. (4.47).

16 In the MCRF, \( U^\beta = 0 \). Why can’t we assume \( U^\beta = 0 \)?
We have defined \( a^\mu = U^\mu,\beta U^\beta \). Go to the nonrelativistic limit (small velocity) and show that
\[
a^i = \dot{v}^i + (v \cdot \nabla)v^i = Dv^i / Dt,
\]
where the operator \( D/Dt \) is the usual ‘total’ or ‘advective’ time derivative of fluid dynamics.

Sharpen the discussion at the end of § 4.6 by showing that \( -\nabla p \) is actually the net force per unit volume on the fluid element in the MCRF.

Show that Eq. (4.58) can be used to prove Gauss’ law, Eq. (4.57).

(a) Show that if particles are not conserved but are generated locally at a rate \( \varepsilon \) particles per unit volume per unit time in the MCRF, then the conservation law, Eq. (4.35), becomes
\[
N^\alpha_{,\alpha} = \varepsilon.
\]
(b) Generalize (a) to show that if the energy and momentum of a body are not conserved (e.g. because it interacts with other systems), then there is a nonzero relativistic force four-vector \( F^\alpha \) defined by
\[
T^\alpha_{\beta,\beta} = F^\alpha.
\]
Interpret the components of \( F^\alpha \) in the MCRF.

In an inertial frame \( O \) calculate the components of the stress–energy tensors of the following systems:

(a) A group of particles all moving with the same velocity \( v = \beta e_x \), as seen in \( O \). Let the rest-mass density of these particles be \( \rho_0 \), as measured in their comoving frame. Assume a sufficiently high density of particles to enable treating them as a continuum.

(b) A ring of \( N \) similar particles of mass \( m \) rotating counter-clockwise in the \( x - y \) plane about the origin of \( O \), at a radius \( a \) from this point, with an angular velocity \( \omega \). The ring is a torus of circular cross-section of radius \( \delta a \ll a \), within which the particles are uniformly distributed with a high enough density for the continuum approximation to apply. Do not include the stress–energy of whatever forces keep them in orbit. (Part of the calculation will relate \( \rho_0 \) of part (a) to \( N, a, \omega, \) and \( \delta a \).)

(c) Two such rings of particles, one rotating clockwise and the other counter-clockwise, at the same radius \( a \). The particles do not collide or interact in any way.

Many physical systems may be idealized as collections of noncolliding particles (for example, black-body radiation, rarified plasmas, galaxies, and globular clusters). By assuming that such a system has a random distribution of velocities at every point, with no bias in any direction in the MCRF, prove that the stress–energy tensor is that of a perfect fluid. If all particles have the same speed \( v \) and mass \( m \), express \( p \) and \( \rho \) as functions of \( m, v, \) and \( n \). Show that a photon gas has \( p = \frac{1}{3} \rho \).

Use the identity \( T^{\mu\nu,\nu} = 0 \) to prove the following results for a bounded system (i.e. a system for which \( T^{\mu\nu} = 0 \) outside a bounded region of space):
4.10 Exercises

(a) \( \frac{\partial}{\partial t} \int T^{00} \, d^3x = 0 \) (conservation of energy and momentum).

(b) \( \frac{\partial^2}{\partial t^2} \int T^{00} x^i x^j \, d^3x = 2 \int T^{ij} x^i \, d^3x \) (tensor virial theorem).

(c) \( \frac{\partial^2}{\partial t^2} \int T^{00} (x^i x^j)^2 \, d^3x = 4 \int T^{i0} x^i \, d^3x + 8 \int T^{ij} x^i x^j \, d^3x. \)

24 Astronomical observations of the brightness of objects are measurements of the flux of radiation \( T^{0i} \) from the object at Earth. This problem calculates how that flux depends on the relative velocity of the object and Earth.

(a) Show that, in the rest frame \( \mathcal{O} \) of a star of constant luminosity \( L \) (total energy radiated per second), the stress–energy tensor of the radiation from the star at the event \((t, x, 0, 0)\) has components \( T^{00} = T^{0x} = T^{x0} = T^{xx} = L/(4\pi x^2) \). The star sits at the origin.

(b) Let \( \mathbf{X} \) be the null vector that separates the events of emission and reception of the radiation. Show that \( \mathbf{X} \rightarrow \mathcal{O}(x, x, 0, 0) \) for radiation observed at the event \((x, x, 0, 0)\). Show that the stress–energy tensor of (a) has the frame-invariant form

\[
\mathbf{T} = \frac{L}{4\pi} \left( \mathbf{X} \otimes \mathbf{X} \right),
\]

where \( \mathbf{U}_s \) is the star’s four-velocity, \( \mathbf{U}_s \rightarrow \mathcal{O}(1, 0, 0, 0) \).

(c) Let the Earth-bound observer \( \mathbf{O} \), traveling with speed \( \nu \) away from the star in the \( x \) direction, measure the same radiation, again with the star on the \( \mathbf{x} \) axis. Let \( \mathbf{X} \rightarrow (R, R, 0, 0) \) and find \( R \) as a function of \( x \). Express \( T^{0i} \) in terms of \( R \). Explain why \( R \) and \( T^{0i} \) depend as they do on \( \nu \).

25 Electromagnetism in SR. (This exercise is suitable only for students who have already encountered Maxwell’s equations in some form.) Maxwell’s equations for the electric and magnetic fields in vacuum, \( \mathbf{E} \) and \( \mathbf{B} \), in three-vector notation are

\[
\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E} = 4\pi \mathbf{J}, \\
\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} = 0,
\]

in units where \( \mu_0 = \varepsilon_0 = c = 1 \). (Here \( \rho \) is the density of electric charge and \( \mathbf{J} \) the current density.)

(a) An antisymmetric \( (2) \) tensor \( \mathbf{F} \) can be defined on spacetime by the equations \( F^{0i} = E^i (i = 1, 2, 3), F^{xy} = B^z, F^{zx} = B^x, F^{zx} = B^z \). Find from this definition all other components \( F^{\mu\nu} \) in this frame and write them down in a matrix.

(b) A rotation by an angle \( \theta \) about the \( z \) axis is one kind of Lorentz transformation, with the matrix

\[
\Lambda_{\beta^\prime}^\alpha = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Show that the new components of $\mathbf{F}$,

$$F^{\alpha' \beta'} = \Lambda^\alpha_\mu \Lambda^\beta_\nu F^{\mu \nu},$$

define new electric and magnetic three-vector components (by the rule given in (a)) that are just the same as the components of the old $\mathbf{E}$ and $\mathbf{B}$ in the rotated three-space. (This shows that a spatial rotation of $\mathbf{F}$ makes a spatial rotation of $\mathbf{E}$ and $\mathbf{B}$.)

(c) Define the current four-vector $\mathbf{J}$ by $J^0 = \rho, J^i = (\mathbf{J})^i$, and show that two of Maxwell’s equations are just

$$F^{\mu \nu} = 4\pi J^\mu.$$  (4.60)

(d) Show that the other two of Maxwell’s equations are

$$F_{\mu \nu, \lambda} + F_{\nu \lambda, \mu} + F_{\lambda \mu, \nu} = 0.$$  (4.61)

Note that there are only four independent equations here. That is, choose one index value, say zero. Then the three other values (1, 2, 3) can be assigned to $\mu, \nu, \lambda$ in any order, producing the same equation (up to an overall sign) each time. Try it and see: it follows from antisymmetry of $F_{\mu \nu}$.

(e) We have now expressed Maxwell’s equations in tensor form. Show that conservation of charge, $J^\mu, \mu = 0$ (recall the similar Eq. (4.35) for the number–flux vector $\tilde{N}$), is implied by Eq. (4.60) above. (Hint: use antisymmetry of $F_{\mu \nu}$.)

(f) The charge density in any frame is $J^0$. Therefore the total charge in spacetime is $Q = \int J^0 \, dx \, dy \, dz$, where the integral extends over an entire hypersurface $t =$ const. Defining $\mathbf{\tilde{n}} = \hat{n}$, a unit normal for this hyper-surface, show that

$$Q = \int J^\mu n_\alpha \, dx \, dy \, dz.$$  (4.62)

(g) Use Gauss’ law and Eq. (4.60) to show that the total charge enclosed within any closed two-surface $S$ in the hypersurface $t =$ const. can be determined by doing an integral over $S$ itself:

$$Q = \oint_S F^{0i} n_i \, dS = \oint_S \mathbf{E} \cdot \mathbf{n} \, dS,$$

where $\mathbf{n}$ is the unit normal to $S$ in the hypersurface (not the same as $\tilde{n}$ in part (f) above).

(h) Perform a Lorentz transformation on $F^{\mu \nu}$ to a frame $\tilde{\mathbf{O}}$ moving with velocity $\mathbf{v}$ in the $x$ direction relative to the frame used in (a) above. In this frame define a three-vector $\tilde{\mathbf{E}}$ with components $\tilde{E}^i = F^{0i}$, and similarly for $\tilde{\mathbf{B}}$ in analogy with (a). In this way discover how $\mathbf{E}$ and $\mathbf{B}$ behave under a Lorentz transformation: they get mixed together! Thus, $\mathbf{E}$ and $\mathbf{B}$ themselves are not Lorentz invariant, but are merely components of $\mathbf{F}$, called the Faraday tensor, which is the invariant description of electromagnetic fields in relativity. If you think carefully, you will see that on physical grounds they cannot be invariant. In particular, the magnetic field is created by moving charges; but a charge moving in one frame may be at rest in another, so a magnetic field which exists in one frame may not exist in another. What is the same in all frames is the Faraday tensor: only its components get transformed.